COMPLEX ANALYSIS AND RIEMANN SURFACES

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1. A review of complex analysis

Preliminaries. The complex numbers may be constructed as $\mathbb{C} = \mathbb{R}[x]/(x^2+1)$. Since they are a field extension of the reals they form a vector space over $\mathbb{R}$ of dimension 2. The Galois group of $\mathbb{C}/\mathbb{R}$ has size two, with the nontrivial element $\sigma : \mathbb{C} \to \mathbb{C}$ interchanging the two adjoined roots of x^2+1. that is $\sigma(i) = -i$ where i is a solution to $x^2+1=0$. The action of σ on an element of $\mathbb{C}$ is usually written $\sigma(z) = \bar{z}$ and called complex conjugation.

Given a vector space V over the complex numbers, we can form the conjugate vector space $\bar{V}$ whose set and addition is identical to that of V but whose scalar multiplication is given by $z * v = \bar{z} \cdot v$, where $\cdot$ is scalar multiplication on V and $*$ that of $\bar{V}$. Focusing on the 1-dimensional complex vector space $\mathbb{C}$, we see that we actually have two copies of the complex plane $\mathbb{C}$ and $\bar{\mathbb{C}}$ depending on which root of x^2+1 we choose to be called i.

Since 1 is a complex basis, 1 and i form a real basis for $\mathbb{C}$. So, for each $z \in \mathbb{C}$ there exists unique x and y in $\mathbb{R}$ so that $z = x+iy$. Multiplication by a complex number $a+ib$ is a linear function given by

$$M(z) = (a+ib)(x+iy) = ax - by + i(ay+bx).$$

Since $M(1) = a+ib$ and $M(i) = -b+ia$ we have the matrix representation of the real linear map M in the basis $(1,i)$ is given by

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Notice that $\det(M) = a^2+b^2 = |a+ib|^2$ implies M is invertible provided $z \neq 0$. This tells us we have an embedding of the multiplicative group of complex numbers into $\mathrm{GL}(2,\mathbb{R})$. Multiplication by i is especially important; it is given by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Lemma. *Let $M : \mathbb{C} \to \mathbb{C}$ be a linear map. Then*

$$M(z) = \alpha z + \beta \bar{z}$$

for two complex numbers α and β.

Proof. Since $M(x+iy) = xM(1) + M(i)y$ and since $x = (z+\bar{z})/2$ and $y = (z-\bar{z})/2i$ we have

$$M(z) = \left(\frac{M(1) - iM(i)}{2}\right) z + \left(\frac{M(1)+iM(i)}{2}\right)\bar{z}.$$

□

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For $M(z) = \alpha z + \beta \bar{z}$,

- if α and β are both nonzero then M is only real linear $\mathbb{C} \to \mathbb{C}$.
- If $\beta = 0$, then M is a complex linear map $\mathbb{C} \to \mathbb{C}$.
- If $\alpha = 0$, then M is an complex antilinear map $\mathbb{C} \to \mathbb{C}$, which is equivalent to being a complex linear map $\mathbb{C} \to \bar{\mathbb{C}}$.

In this sense we say that a linear map is complex linear if it is independent of $\bar{z}$. If M is a complex linear map $\mathbb{C} \to \mathbb{C}$. Then $M(z) = zM(1)$, so M is given by multiplication by the complex number $\alpha = M(1)$. This gives an isomorphism $\mathrm{End}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}$, the map given by $M \mapsto M(1)$. In terms of matrices, M is complex linear if it commutes with multiplication by i:

$$M \circ J = J \circ M.$$

Holomorphic Functions. With the norm $|x+iy| = \sqrt{x^2 + y^2}$, the complex numbers $\mathbb{C}$ become a (real and complex) topological vector space. Let Ω be an open subset and suppose $f : \Omega \to \mathbb{C}$.

Definition. *We say f is differentiable at a point $z \in \Omega$ if there exists a continuous linear map $df_z : \mathbb{C} \to \mathbb{C}$, called the derivative of f at z, such that*

$$\lim_{v \to 0} \frac{f(z+v) - f(z) - df_z(v)}{v} = 0.$$

All normed vector spaces of the same finite dimension are homeomorphic, and so this limit is independent of the chosen norm. Also note, in finite dimensions all linear maps are continuous so this condition is perhaps unnecessary.

Definition. *A function $f : \Omega \to \mathbb{C}$ is called holomorphic at a point z if there is a neighborhood of z in which f is differentiable at each point and each derivative is a complex linear map. The function f is called antiholomorphic if instead the derivative is antilinear at each point.*

If f is antiholomorphic then $\bar{f} = \sigma \circ f$ is holomorphic since $d(\sigma \circ f)_z = d(\sigma)_{f(z)} \circ df_z$ is complex linear.

In the holomorphic case we can compute

$$\frac{f(z+wv) - f(z)}{w} - df_z(v) = \frac{f(z+wv) - f(z) - df_z(wv)}{wv} v \to 0$$

as $w \to 0$ so that the derivative is given by

$$df_z(v) = \lim_{w \to 0} \frac{f(z+wv) - f(z)}{w}.$$

Since the derivative df_z is a complex linear map, it is given by multiplication by a complex number $df_z(1)$. We define $\frac{df}{dz}$ as this number, and it may be computed by

$$\frac{df}{dz} = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w}.$$

In the antiholomorphic case, df_z is antilinear and so a complex linear map $\mathbb{C} \to \bar{\mathbb{C}}$. In this case the above computation becomes

$$\frac{f(z+wv) - f(z)}{\bar{w}} - df_z(v) = \frac{f(z+wv) - f(z) - df_z(wv)}{wv} \frac{wv}{\bar{w}} \to 0$$

as $w \to 0$ so that the derivative is given by

$$df_z(v) = \lim_{w \to 0} \frac{f(z+\bar{w}v) - f(z)}{w}.$$

We consequently define for antiholomorphic functions

$$\frac{df}{d\bar{z}} = \lim_{w\to 0} \frac{f(z+w)-f(z)}{\bar{w}} = \lim_{w\to 0} \frac{f(z+\bar{w})-f(z)}{w}.$$

Since 1 and i are a real basis, we have functions $u, v : \Omega \to \mathbb{R}$ such that $f(z) = u(z) + iv(z)$. Now, suppose f is holomorphic at a point $z \in \Omega$. Then df_z exists and is in particular a real linear map $\mathbb{C} \to \mathbb{C}$. We therefore have $du_z = \operatorname{Re}(df_z)$ and $dv_z = \operatorname{Im}(df_z)$ exist and consequently, the partial derivatives u_x, u_y, v_x, v_y exists at z.

We now find the matrix of this real linear map by computing $df_z(1) = f'(z)$ and $df_z(i)$. For x real

$$df_z(1) = \lim_{x\to 0} \frac{u(z+x)-u(z)}{x} + i\frac{v(z+x)-v(z)}{x} = u_x(z) + iv_x(z).$$

A similar computation for y real gives

$$df_z(i) = \lim_{y\to 0} \frac{u(z+iy)-u(z)}{y} + i\frac{v(z+iy)-v(z)}{y} = u_y(z) + iv_y(z).$$

Consequently, the matrix of the real linear map df_z is given by

$$df_z = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

the partial derivatives evaluated at z. This same matrix is also the representation of an antilinear df_z since the real vector spaces $\mathbb{C}$ and $\bar{\mathbb{C}}$ are the same. In the complex linear case this matrix must commute with multiplication by i, that is, $df_z \circ J = J \circ df_z$. This condition simplifies to

$$\begin{pmatrix} u_y & -u_x \\ v_y & -v_x \end{pmatrix} = \begin{pmatrix} -v_x & -v_y \\ u_x & u_y \end{pmatrix}.$$

When df_z is antilinear we have that df_z anti-commutes with J, i.e, $df_z \circ J = -J \circ df_z$. This implies,

$$\begin{pmatrix} u_y & -u_x \\ v_y & -v_x \end{pmatrix} = \begin{pmatrix} v_x & v_y \\ -u_x & -u_y \end{pmatrix}.$$

In summary we have the following system of equations.

Theorem. *Suppose $f = u + iv$ is differentiable at $z \in \Omega$. If df_z is complex linear then u and v satisfy the* **Cauchy-Riemann equations**

$$u_x = v_y$$
$$u_y = -v_x.$$

If df_z is complex antilinear then u and v satisfy the **Conjugate-Cauchy-Riemann equations**

$$u_x = -v_y$$
$$u_y = v_x.$$

Define $\frac{\partial f}{\partial x} = u_x + iv_x$ and similarly $\frac{\partial f}{\partial y} = u_y + iv_y$. For a differentiable function $f : \Omega \to \mathbb{C}$ we have the real linear map df_z is given by

$$\begin{aligned} df_z(v) &= \left(\frac{df_z(1) - idf_z(i)}{2}\right) v + \left(\frac{df_z(1) + idf_z(i)}{2}\right) \bar{v} \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) v + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \bar{v} \end{aligned}$$

Now if f is holomorphic then

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{df}{dz}$$

by the Cauchy-Riemann equations, while

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

Analogously, using the Conjugate-Cauchy-Riemann equations we have

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{df}{d\bar{z}} \quad \text{and} \quad \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = 0.$$

Using this as a suggestion have the following definition

Definition. *On the set of smooth complex valued functions $\Omega \to \mathbb{C}$ we define the partial derivative opeartors*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \textit{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Theorem. *Suppose $f : \Omega \to \mathbb{C}$ is smooth. Then*

$$df_z(v) = \frac{\partial f}{\partial z} v + \frac{\partial f}{\partial \bar{z}} \bar{v}.$$

When f is holomorphic then $\frac{\partial f}{\partial z} = \frac{df}{dz}$ and $\frac{\partial f}{\partial \bar{z}} = 0$. When f is antiholomorphic then $\frac{\partial f}{\partial \bar{z}} = \frac{df}{d\bar{z}}$ and $\frac{\partial f}{\partial z} = 0$.

The Complex Exponential. What should the meaning of e^z be for z a complex number? Whatever it means, we should want it to still behave as a homomorphism $e^{z+w} = e^z e^w$ from the additive group of complex numbers to the multiplicative group of complex numbers. And, we should want that it reduce to the ordinary exponential when z is real. And really, it's not too much to ask that it be holomorphic as well since the real exponential is smooth.

Let's assume these stipulations. If we write $z = x + iy$ then we get $e^z = e^{x+iy} = e^x e^{iy}$. Since we're asking that e^x be the ordinary exponential, we really just need to determine what e^{iy} should be. To do this we will use a property of holomorphic functions on $\mathbb{C}$ which are extensions of smooth functions of $\mathbb{R}$.

Suppose F is a holomorphic extension of a smooth function $\mathbb{R} \to \mathbb{R}$. This means that there is a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $F|_{\mathbb{R}} = f$. Now, if z is real, then $F(z) = f(z)$ is real as well. So we have $\bar{z} = z$ implies $\overline{F(z)} = F(z)$. Or equivalently,

$$z \in \mathbb{R} \implies F(z) = \overline{F(\bar{z})}.$$

As it turns out, this equality is true for every $z \in \mathbb{C}$ when F is holomorphic. This follows from the Identity Theorem.

Theorem. *Suppose f and g are two holomorphic function defined on an open and connected set Ω. Then if $f(z) = g(z)$ for all z in some set which has an accumulation point, then $f = g$ on Ω.*

Note that $\overline{F(\bar{z})} = (\sigma \circ F \circ \sigma)(z)$, where σ is complex conjugation, is holomorphic since σ itself being linear and antiholomorphic implies that the derivative $d(\sigma \circ F \circ \sigma)_z$ is complex linear. Now, since $\mathbb{R}$ has an accumulation point, and since $F(z) = \overline{F(\bar{z})}$ on $\mathbb{R}$, by the Identity Theorem we get that $F(z) = \overline{F(\bar{z})}$ on $\mathbb{C}$. Or written more usefully, $F(\bar{z}) = \overline{F(z)}$.

Applying this to the complex exponential, we have $\overline{e^{iy}} = e^{-iy}$ and therefore, $|e^{iy}|^2 = e^{iy}e^{-iy} = 1$. Consequently,

$$e^{iy} = \cos(\theta(y)) + i\sin(\theta(y))$$

for some function θ of y. By the Inverse Function Theorem applied to cosine, we get that θ is smooth away from y such that $\theta(y) = k\pi$. But the same reasoning applied to sine gives that θ is smooth at y with $\theta(y) = k\pi$. Hence θ is a smooth function of y.

We have show that for $z = x + iy$ the holomorphic function e^z can be written as

$$e^{x+iy} = e^x \cos(\theta(y)) + ie^x \sin(\theta(y))$$

for a smooth function θ. By the Cauchy-Riemann equations we have

$$e^x \cos(\theta(y)) = e^x \cos(\theta(y))\theta'(y) \quad \text{and} \quad -e^x \sin(\theta(y))\theta'(y) = -e^x \sin(\theta(y)).$$

Both of these together tell us that $\theta'(y) = 1$. Consequently, $\theta(y) = y + c$ for some constant c. To find this constant note that

$$1 = e^0 = \cos(c) + i\sin(c),$$

so that $c = 2\pi k$ for some integer k. But sine and cosine are 2π periodic so we may take $k = 0$.

Theorem (Euler's Formula)**.** *The complex exponential is the holomorphic function with sends $z = x + iy$ to*

$$e^{x+iy} = e^x(\cos(y) + i\sin(y)).$$

Note that due to the periodicity of the trig functions, the complex exponential is not injective since $e^z = e^{z+2\pi ik}$. Hence we cannot expect to have a globally defined complex logarithm on $\mathbb{C} - \{0\}$.

Contour Integration. The contour integral of a function $f : \Omega \to \mathbb{C}$ over a differentiable curve $\gamma : (a, b) \to \Omega$ is defined as

$$\int_\gamma f \; dz = \int_a^b f(\gamma(t))\gamma'(t) \; dt.$$

Writing $f(z) = u(z) + iv(z)$ and $\gamma(t) = x(t) + iy(t)$ we can compute

$$\begin{aligned} f(\gamma(t))\gamma'(t) &= (u + iv)(x' + iy') \\ &= ux' - vy' + i(uy' + vx') \\ &= (u - iv) \cdot (x' + iy') + i(u - iv) \cdot (y' - ix'). \end{aligned}$$

If T and n are the unit tangent and normal vectors to γ, then a computation gives

$$(\bar{f} \cdot T)|\gamma'| = (u - iv) \cdot (x' + iy')$$
$$(\bar{f} \cdot n)|\gamma'| = (u - iv) \cdot (y' - ix').$$

Hence, the contour integral of f along γ is measuring both the circulation and the flux of the vector field $\bar{f}$ over γ:

$$\int_\gamma f(z)dz = \int_\gamma \bar{f} \cdot T \; ds + i \int_\gamma \bar{f} \cdot n \; ds.$$

As an example, suppose $f(z) = \frac{1}{z}$. Then in terms of the real and imaginary components this is

$$f(x + iy) = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2},$$

so the conjugate is

$$\bar{f}(x + iy) = \frac{x}{x^2 + y^2} + i\frac{y}{x^2 + y^2}.$$

When γ is the unit circle, this radial vector field is orthogonal to the unit tangent vector field T and equal to unit normal n. Hence $\bar{f} \cdot T = 0$ and $\bar{f} \cdot n = 1$. Thus,

$$\int_\gamma \frac{1}{z} \; dz = 2\pi i.$$

Theorem. *Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and C^1 and that γ is a simple closed curve in Ω that is the boundary of a simply connected disk D. Then the integral of f over γ is*

$$\int_\gamma f(z) \; dz = 0.$$

Proof. By Green's Theorem we get

$$\begin{aligned}\int_\gamma f(z) \; dz &= \int_\gamma \bar{f} \cdot T \; ds + i \int_\gamma \bar{f} \cdot n \; ds \\ &= \iint_D -(v_x + u_y) \; dA + i \iint_D (u_x - v_y) \; dA,\end{aligned}$$

and these last two integrals vanish by the Cauchy Riemann equations. □

2. Riemann Surfaces

2.1. **The Holomorphic Tangent Space.**

A Riemann surface is a 1 dimensional complex manifold. Specifically

Definition. *A Riemann Surface is a topological space S with an atlas of charts $\varphi_j : U_j \to \mathbb{C}$ whose transition functions $\varphi_k \circ \varphi_j^{-1}$ are holomorphic.*

Since holomorphic functions are smooth, a Riemann surface structure on S also gives a smooth surface structure on S. Let X denote the pair S together with a given holomorphic atlas. Some examples include the Riemann sphere, projective space, and open subsets of the plane.

A map $F : X \to Y$ between two Riemann surfaces is called holomorphic if $\psi \circ F \circ \varphi^{-1}$ is holomorphic for all admissible charts φ on X and ψ on Y. We want to define tangent vectors to a Riemann surface the same way we defined real tangent vectors to a smooth surface. That is we want to define them as derivations of holomorphic functions. The issue is that for compact Riemann surfaces there are no non-constant global holomorphic functions. We need to work with the sheaf of holomorphic functions instead.

The sheaf of holomorphic functions on X is denoted by $\mathcal{O}$. So for each open set Ω in X, the set $\mathcal{O}(\Omega)$ is the set of holomorphic functions $\Omega \to \mathbb{C}$. The stalk of $\mathcal{O}$ at a point $p \in X$ is

$$\mathcal{O}_p = \varinjlim \mathcal{O}(\Omega),$$

which can be thought of as germs of holomorphic functions at p. The stalk $\mathcal{O}_p$ is a complex algebra and so we define the tangent space at p to X to be the complex derivations of this algebra. That is

Definition. *The holomorphic tangent space to X at p, denoted by T_pX, is the set of all derivations $v : \mathcal{O}_p \to \mathbb{C}$. That is, all complex linear maps v such that $v(fg) = v(f)g(p) + f(p)v(g)$. Note that each function belonging to a point of $\mathcal{O}_p$ takes the same value at p and so this abuse of notation is allowed.*

If F is a holomorphic function from X to Y, its derivative at $p \in X$ is a complex linear map between the tangent space $dF_p : T_pX \to T_{F(p)}Y$ where the image of a vector v acts on the stalk of holomorphic functions at $F(p)$ by

$$dF_p(v)f = v(f \circ F).$$

Given a point $p \in \mathbb{C}$, the operator $\frac{d}{dz}$ is a tangent vector to $\mathbb{C}$ at p, its action given by

$$\frac{d}{dz}f = f'(p) = \lim_{z \to 0} \frac{f(p+z) - f(p)}{z}.$$

In fact, this vector spans the tangent space $T_p\mathbb{C}$. This follows since holomorphic functions are analytic: suppose $v \in T_p\mathbb{C}$, then

$$v(f) = v\left(\sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n\right) = f'(p)v(z-p) = v(z)\frac{d}{dz}f,$$

so $v = v(z)\frac{d}{dz}$ and we see $T_p\mathbb{C}$ is one dimensional. By sending v to $v(z)$ we also get a canonical identification of $T_p\mathbb{C}$ with $\mathbb{C}$ itself.

Now, a chart $z : U \to \mathbb{C}$ on X gives an identification of T_pX with $T_{z(p)}\mathbb{C}$ and therefore, we can send $\frac{d}{dz}$ to X to get a tangent vector $\frac{d}{dz} \in T_pX$ that acts by

$$\frac{d}{dz}f := d(z^{-1})_{z(p)}\left(\frac{d}{dz}\right)f = \frac{d}{dz}(f \circ z^{-1}) = (f \circ z^{-1})'(z(p)),$$

i.e., by taking the derivative of the coordinate representation of f at the coordinate representation of p. Again, this vector is a basis for T_pX. In particular, thinking of a chart z as a holomorphic function we see that

$$\frac{d}{dz}z = (z \circ z^{-1})'(z(p)) = \lim_{w \to 0} \frac{z(p) + w - z(p)}{w} = 1,$$

as expected.

The holomorphic cotangent space is defined to be the dual space to the holomorphic tangent space $T_p^*X = (T_pX)^*$. This is the space of all complex linear functionals $T_pX \to \mathbb{C}$. Examples include the derivative of any function holomorphic in a neighborhood of p. That is, if f is holomorphic around p then $df_p : T_pX \to T_{f(p)}\mathbb{C} \cong \mathbb{C}$ so $df_p \in T_p^*X$. More explicitly, if $v \in T_pX$ and f is holomorphic near p then then number $df_p(v) \in \mathbb{C}$ is

$$df_p(v) \sim df_p(v)(z) = v(z \circ f) = v(f)$$

So if we always identify the tangent spaces to $\mathbb{C}$ itself then for any holomorphic function defined near $p \in X$ we have $df_p(v) = v(f)$. If z is a chart around p then we can think of z as a holomorphic function near p and so dz_p is a cotangent vector. It sends the basis vector $\frac{d}{dz}$ to

$$dz_p\left(\frac{d}{dz}\right) = \frac{d}{dz}z = 1.$$

Since the holomorphic tangent space is one dimensional so too is the cotangent space. Moreover, dz_p is a basis.

2.2. **The Real Tangent Space.** The Riemann surface X has an underlying smooth surface structure S and therefore already has a notion of tangent and cotangent spaces at a point. Our main interest is how these relate to the just defined holomorphic tangent and cotangent spaces.

Let $z : U \to \mathbb{C}$ be a holomorphic chart. Then writing $z = x + iy$ we get two real valued functions $x, y : U \to \mathbb{R}$ smooth on the surface S. Moreover, z provides a real coordinated chart for S if we interpret it as taking values in the real vector space $\mathbb{C}$.

Let $\mathcal{C}^\infty_\mathbb{R}$ be the sheaf of smooth real valued functions on S. So, $\mathcal{C}^\infty_\mathbb{R}(\Omega) = C^\infty(\Omega, \mathbb{R})$ for Ω an open subset of S. Denoting by $(\mathcal{C}^\infty_\mathbb{R})_p$ the stalk of this sheaf at p we have the real tangent space to S at p is all real derivations of this real algebra.

Definition. *The real tangent space T_pS to S at p is the vector space of all derivations $(\mathcal{C}^\infty_\mathbb{R})_p \to \mathbb{R}$. This is, all real linear maps that satisfy the product rule.*

If z is a holomorphic coordinate chart around p then $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are real tangent vectors at p and span T_pS. The real cotangent space is $T_p^*S = (T_pS)^*$, the space of real linear maps $T_pS \to \mathbb{R}$. It is spanned by dx and dy. Our main goal is to determine how the real and holomorphic tangent and cotangent spaces are related.

We cannot hope that the underlying real vector space of T_pX be equal to T_pS since they consists of derivations of different algebras. If $v \in T_pX$ then $v : \mathcal{O}_p \to \mathbb{C}$ while if $v \in T_pS$ then $v : (\mathcal{C}^\infty_\mathbb{R})_p \to \mathbb{R}$. However, we may extend the action of $v \in T_pS$ to $\mathcal{O}_p$ by linearity. That is, if $f + ig \in \mathcal{O}_p$ then we can define $v(f+ig) = v(f)+iv(g)$. Asking for this extension to be a complex linear map is equivalent to taking the complexification of T_pS.

Consider $T_pS \otimes \mathbb{C}$. This is a complex vector space of dimension $\dim_\mathbb{C}(T_pS \otimes \mathbb{C}) = \dim_\mathbb{R}(T_pS) = 2\dim_\mathbb{C}(T_pX)$. There is a useful isomorphism

$$T_pS \otimes \mathbb{C} = \mathrm{Der}_\mathbb{R}(\mathcal{C}^\infty_\mathbb{R})_p \otimes \mathbb{C} \cong \mathrm{Der}_\mathbb{C}(\mathcal{C}^\infty_\mathbb{C})_p$$

of the complexified real derivations with complex derivations of the stalk of the sheaf $\mathcal{C}^\infty_\mathbb{C}$ of smooth complex valued functions on S, where $v \otimes z$ acts on $f + ig \in (\mathcal{C}^\infty_\mathbb{C})_p$ by

$$(v \otimes z)(f + ig) = z(v(f) + iv(g)).$$

Since holomorphic functions near p are also smooth functions near p we have the inclusion $\mathcal{O}_p \hookrightarrow (\mathcal{C}^\infty_\mathbb{C})_p$ and thus a map $\varphi : T_pS \otimes \mathbb{C} \to T_pX$ by restricting the derivation. Note that

$$T_pS \otimes \mathbb{C}/\ker\varphi = \mathrm{im}\varphi.$$

Therefore, we can investigate this map by finding a decomposition of $T_pS \otimes \mathbb{C}$ that helps us understand this quotient. It turns out that the complex manifold structure

X on S induces an almost complex structure J on S with respect to which we can decompose $T_pS \otimes \mathbb{C}$ into eigenspaces.

2.3. **The Almost Complex Structure.** Recall that $J : \mathbb{C} \to \mathbb{C}$ is the real linear map given by multiplication by i. In particular it satisfies $J^2 = -Id$, so it is an almost complex structure on the real vector space $\mathbb{C}$. Under the identifications of the real tangent spaces to $\mathbb{C}$ with the real vector space $\mathbb{C}$ itself, we have that J gives an almost complex structure on the smooth surface $\mathbb{C}$.

Let $p \in S$ and z a holomorphic coordinate around p. Then $d(z^{-1})_{z(p)}$ gives a real isomorphism $\mathbb{C} \cong T_{z(p)}\mathbb{C} \to T_pS$. So, we can pull back the almost complex structure J on $T_{z(p)}\mathbb{C}$ to T_pS using this chart. That is, define the almost complex structure $\tilde{J}$ on T_pS by

$$\tilde{J} = d(z^{-1})_{z(p)} \circ J_{z(p)} \circ dz_p.$$

This almost complex structure is independent of the chosen holomorphic coordinate chart specifically since the chart is holomorphic. Indeed, suppose w is another holomorphic chart around p and denote by $\tilde{J}_z$ and $\tilde{J}_w$ the almost complex structures on T_pS defined as above. Then since the transition function $w \circ z^{-1}$ is holomorphic we have that $d(w \circ z^{-1})_{z(p)}$ is complex linear and so $d(w \circ z^{-1})_{z(p)}$ thought of as a real linear map commutes with J; i.e, $d(w \circ z^{-1})_{z(p)} \circ J_{z(p)} = J_{w(p)} \circ d(w \circ z^{-1})_{z(p)}$. Hence,

$$\begin{aligned}
\tilde{J}_w &= d(w^{-1})_{w(p)} \circ J_{w(p)} \circ dw_p \\
&= d(w^{-1})_{w(p)} \circ d(w \circ z^{-1})_{z(p)} \circ J_{z(p)} \circ d(z \circ w^{-1})_{w(p)} \circ dw_p \\
&= d(z^{-1})_{z(p)} \circ J_{z(p)} \circ dz_p \\
&= \tilde{J}_z.
\end{aligned}$$

And so we have a well defined almost complex structure on T_pS that is induced by the complex structure X on S.

More specifically we have $J : T_pS \to T_pS$ is real linear and $J^2 = -Id$. So while J has no real eigenvalues as a map on T_pS, we can extend it to the complexification $J : T_pS \otimes \mathbb{C} \to T_pS \otimes \mathbb{C}$ where it will have eigenvalues $\pm i$. Therefore, we can write

$$T_pS \otimes \mathbb{C} = T_p^{1,0}X \oplus T_p^{0,1}X,$$

where $T_p^{1,0}X$ is the $+i$ eigenspace and $T_p^{0,1}X$ is the $-i$ eigenspace. The direct sum decomposition is given by

$$v = \frac{1}{2}(v - iJ(v)) + \frac{1}{2}(v + iJ(v)).$$

Consider the inclusion $(T_pS, J) \hookrightarrow T_pS \otimes \mathbb{C}$. Then $J(v)$ maps to

$$J(v) \mapsto i\frac{1}{2}(v - iJ(v)) + (-i)\frac{1}{2}(v + iJ(v)).$$

So we see that the map $(T_pS, J) \to (T_p^{1,0}X, i)$ is a complex linear isomorphism (by counting dimensions).

2.4. **The Identifications.** Now choose a holomorphic coordinate z near p, which in turn induces real coordinates (x, y) near p. Then if we follow J through the identification $T_{z(p)}\mathbb{C} \cong \mathbb{C}$ we see that $J\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}$. Then under the isomorphism just discussed, the *complex basis* of (T_pS, J), $\frac{\partial}{\partial x}$, maps to

$$\frac{\partial}{\partial x} \mapsto \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

in $T_p^{1,0}X$. This is then a complex basis for $(T_p^{1,0}X, i)$. We suggestively define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

What is the image of $\frac{\partial}{\partial z}$ under the restriction map $T_pS \otimes \mathbb{C} \to T_pX$? Let $f = u + iv$ be holomorphic near p. Then

$$\begin{aligned}\frac{\partial}{\partial z}f &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv)\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right)\\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{d}{dz}f\end{aligned}$$

by the Cauchy-Riemann equations. So we see that $\frac{\partial}{\partial z} = \frac{d}{dz}$ on holomorphic functions. In particular, the restriction map $T_pS \otimes \mathbb{C} \to T_pX$ is surjective, with $T_p^{1,0}X \to T_pX$ a complex isomorphism.

What is the kernel of this map? The basis vector $\frac{\partial}{\partial x}$ projects to

$$\frac{\partial}{\partial x} \mapsto \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

in $T_p^{0,1}X$, which we define as

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

This is a real linear isomorphism $T_pS \cong T_p^{0,1}X$ and a complex anti-linear isomorphism $(T_pS, J) \cong (T_p^{0,1}X, i)$. What is the image of $\frac{\partial}{\partial \bar{z}}$ under the restriction $T_pS \otimes \mathbb{C} \to T_pX$? Again, let $f = u + iv$ be holomorphic near p. Then

$$\frac{\partial}{\partial \bar{z}}f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) = 0.$$

That is, $\frac{\partial}{\partial \bar{z}}$ spans the kernel of the restriction map.

What do real tangent vectors $v \in T_pS$ look like under the inclusion $T_pS \hookrightarrow T_pS \otimes \mathbb{C}$? Let $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$. Then

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \mapsto (a+ib)\frac{\partial}{\partial z} + (a-ib)\frac{\partial}{\partial \bar{z}}.$$

Consequenlty, real tangent vectors in $T_pS \otimes \mathbb{C}$ are of the form $v = w\frac{\partial}{\partial z} + \bar{w}\frac{\partial}{\partial \bar{z}}$ for some complex number w. And, under the identification $T_pX \cong T_p^{1,0}X$, holomorphic tangent vectors are of the form $w\frac{\partial}{\partial z}$.

Put all together we get the following.

Theorem. *The vector space* $T_p^{1,0}X$ *is complex isomorphic to* T_pX *via the natural restriction map. Moreover, in a holomorphic coordinate chart* z *around* p *we have* $\frac{\partial}{\partial z}$ *when restricted to holomorphic functions near* p *is equal to* $\frac{d}{dz}$*. The vector* $\frac{\partial}{\partial \bar{z}}$ *annihilates holomorphic functions near* p*; i.e.,* $\ker \frac{\partial}{\partial \bar{z}} = \mathcal{O}_p$.

Since z is holomorphic, we get $\frac{\partial}{\partial z} z = \frac{d}{dz} z = 1$ and $\frac{\partial}{\partial \bar{z}} z = 0$, as expected. The conjugate function $\bar{z}$ is not holomorphic, so we compute

$$\frac{\partial}{\partial z}\bar{z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x - iy) = 0$$

and

$$\frac{\partial}{\partial \bar{z}}\bar{z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x + iy).$$

This justifies the notation for $\frac{\partial}{\partial \bar{z}}$. Because of these identifications, we may use complex notation even when the objects we are considering are real. We can do this by making the substitutions $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$. Then functions of (x, y) we may regard as functions of $(z, \bar{z})$. For example, $\mathrm{Re}(x + iy) = x$ we may regard as $\mathrm{Re}(z, \bar{z}) = \frac{1}{2}(z + \bar{z})$. Then, by identifying $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ with $(a + ib)\frac{\partial}{\partial z} + (a - ib)\frac{\partial}{\partial \bar{z}}$, the action of v on Re can be computed by

$$v(\mathrm{Re}) = \left((a + ib)\frac{\partial}{\partial z} + (a - ib)\frac{\partial}{\partial \bar{z}}\right)\left(\frac{1}{2}(z + \bar{z})\right) = a.$$

as expected from $\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)x = a$.